TEMPERATURE DISTRIBUTION IN A ROD WITH TEMPERATURE OSCILLATIONS ON ITS SURFACE

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A solution is found for the temperature distribution in a semi-infinite rod with damped oscillations of temperature on its surface, allowing for heat transfer with the surrounding medium. Some special cases—no damping, no heat transfer—are examined.

Problems of heat conduction of a solid body with a periodically varying surface temperature are a matter of great practical interest. They are encountered in experimental determination of thermal conductivity, in calculation of the periodically varying temperatures on the walls of internal combustion engine cylinders, and so on [1].

Carslaw and Jaeger [1] have solved the problems of temperature oscillations in a semi-infinite solid whose surface temperature is a harmonic function of time. The present paper examines the more general case of propagation of temperature oscillations in a rod with damped surface temperature oscillations.

We shall examine a semi-infinite rod with a cross-sectional area so small that the temperature is uniform throughout the section. Thus, the problem reduces to one of linear heat flow. On the lateral walls of the rod there is heat transfer with the medium at zero temperature. At the rod surface (x = 0) there are damped temperature oscillations of the type

$$\vartheta(0,\ t) = \vartheta_0 \exp(-kt)\cos(\omega t + \alpha). \tag{1}$$

The equation for the temperature distribution in the rod will have the form

$$\frac{\partial \vartheta(x,\ t)}{\partial t} = a\,\frac{\partial^2 \vartheta(x,\ t)}{\partial x^2} - \beta\vartheta(x,\ t), \tag{2}$$

where

$$\beta = HP/\rho cS.$$

The boundary conditions are

$$\begin{gathered}\vartheta(x, 0) = 0,\\ \vartheta(0,\ t) = \vartheta_0 \exp(-kt)\cos(\omega t + \alpha),\\ \vartheta(\infty\ t) = 0.\end{gathered} \tag{3}$$

We shall solve this problem using Duhamel's theorem; if the temperature distribution in the body, F(x, t) is known when the surface temperature is unity, then the temperature distribution in the case in which the surface temperature is $\varphi(t)$ is given by

$$\vartheta(x,\ t) = \int_0^t \varphi(\lambda)\,\frac{\partial}{\partial t}\,F(x,\ t-\lambda)\,d\lambda. \tag{4}$$

According to [2], for the given problem

$$F(x,\ t) = \frac{1}{2}\left[\exp\left(-\sqrt{\frac{\beta}{a}}\,x\right)\operatorname{erfc}\left(\frac{x}{2\sqrt{at}} - \sqrt{\beta t}\right) + \exp\left(\sqrt{\frac{\beta}{a}}\,x\right)\operatorname{erfc}\left(\frac{x}{2\sqrt{at}} + \sqrt{\beta t}\right)\right], \tag{5}$$

where

$$\operatorname{erfc} u = 1 - \operatorname{erf} u = 1 - \frac{2}{\sqrt{\pi}}\int_0^u \exp(-z^2)\,dz.$$

Determining $(\partial/\partial t)\,F(x,\ t-\lambda)$ and substituting it in (4), we obtain, after some transformations,

$$\vartheta(x,\ t) = \vartheta_0 \int_0^t \exp(-k\lambda)\cos(\omega\lambda + \alpha) \times \frac{x}{2\sqrt{\pi a (t-\lambda)^3}}\exp\left[-\frac{x^2}{4a(t-\lambda)} - \beta(t-\lambda)\right]d\lambda. \tag{6}$$

Introducing the new variable of integration

$$\mu = x/2\sqrt{a(t-\lambda)},$$

we find

$$\vartheta(x,\ t) = \vartheta_0\,\frac{2}{\sqrt{\pi}}\exp(-kt)\int_{\frac{x}{2\sqrt{at}}}^{\infty}\exp\left[\frac{kx^2}{4a\mu^2} - \frac{\beta x^2}{4a\mu^2} - \mu^2\right]\cos\left(\omega t + \alpha - \frac{\omega x^2}{4a\mu^2}\right)d\mu = \vartheta_0\frac{2}{\sqrt{\pi}}\exp(-kt)\int_{\frac{x}{2\sqrt{at}}}^{\infty} R(x,\ t,\ \mu)\,d\mu. \tag{7}$$

We write (7) in the form

$$\vartheta(x,\ t) = \vartheta_1(x,\ t) - \vartheta_2(x,\ t), \tag{8}$$

where

$$\vartheta_1(x,\ t) = \vartheta_0\,\frac{2}{\sqrt{\pi}}\exp(-kt)\int_0^{\infty} R(x,\ t,\ \mu)\,d\mu, \tag{9}$$

$$\vartheta_2(x,\ t) = \vartheta_0\,\frac{2}{\sqrt{\pi}}\exp(-kt)\int_0^{\frac{x}{2\sqrt{at}}} R(x,\ t,\ \mu)\,d\mu. \tag{10}$$

It may be seen from comparison of (9) and (10) that as time increases, $\vartheta_2(x, t)$ decreases much faster than $\vartheta_1(x, t)$, and we shall therefore neglect the contribution of this term to the total temperature $\vartheta(x, t)$ in what follows.

Using the relations given in [3],

$$\int_0^\infty \exp\left[-p^2u^2-\frac{q^2}{u^2}\right]_{\sin}^{\cos}\left(c^2u^2+\frac{b^2}{u^2}\right)du=$$

$$=\frac{\sqrt{\pi}}{2r}\exp\left[-2rs\cos(A+B)\right]_{\sin}^{\cos}\left[A+2rs\sin(A+B)\right],$$

where

$$r=\sqrt[4]{c^4+p^4},\quad s=\sqrt[4]{b^4+q^4},$$

$$A=\frac{1}{2}\operatorname{arctg}\frac{c^2}{p^2},\quad B=\frac{1}{2}\operatorname{arctg}\frac{b^2}{q^2},$$

we finally obtain

$$\vartheta_1(x,t)=\vartheta_0\exp\left[-kt-\frac{x}{\sqrt{a}}\sqrt[4]{(\beta-k)^2+\omega^2}\times\right.$$
$$\left.\times\cos\left(\frac{1}{2}\operatorname{arctg}\frac{\omega}{\beta-k}\right)\right]\times$$
$$\times\cos\left[\omega t+\alpha-\frac{x}{\sqrt{a}}\sqrt[4]{(\beta-k)^2+\omega^2}\times\right.$$
$$\left.\times\sin\left(\frac{1}{2}\operatorname{arctg}\frac{\omega}{\beta-k}\right)\right]. \quad (11)$$

We shall examine some special cases. For steady oscillations of the rod surface temperature ($k = 0$) we have

$$\vartheta(x,t)=\vartheta_0\exp\left[-\frac{x}{\sqrt{a}}\sqrt[4]{\beta^2+\omega^2}\cdot\cos\left(\frac{1}{2}\operatorname{arctg}\frac{\omega}{\beta}\right)\right]\times$$
$$\times\cos\left[\omega t+\alpha-\frac{x}{\sqrt{a}}\sqrt[4]{\beta^2+\omega^2}\cdot\right.$$
$$\left.\cdot\sin\left(\frac{1}{2}\operatorname{arctg}\frac{\omega}{\beta}\right)\right]. \quad (12)$$

With $\beta = 0$, i.e., in the case of damped oscillations at the surface of an infinite solid, we obtain

$$\vartheta(x,t)=\vartheta_0\exp\left\{-kt-\frac{x}{\sqrt{a}}\sqrt[4]{k^2+\omega^2}\times\right.$$
$$\left.\times\cos\left[\frac{1}{2}\operatorname{arctg}\left(-\frac{\omega}{k}\right)\right]\right\}\times\cos\left\{\omega t+\alpha-\right.$$
$$\left.-\frac{x}{\sqrt{a}}\sqrt[4]{k^2+\omega^2}\cdot\sin\left[\frac{1}{2}\operatorname{arctg}\left(-\frac{\omega}{k}\right)\right]\right\}. \quad (13)$$

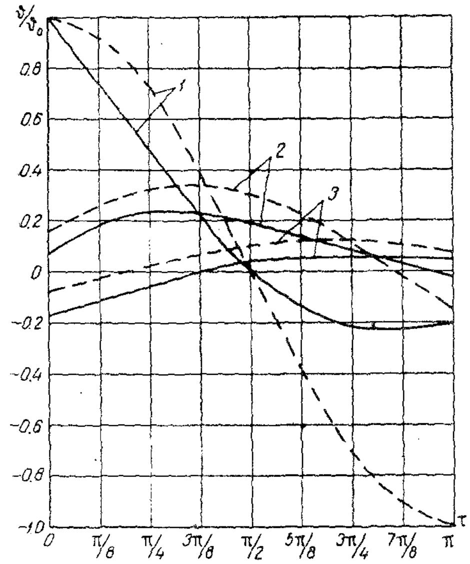
When $k = 0$ and $\beta = 0$, formula (11) goes over to the formula obtained in [1], for the propagation of heat in a semi-infinite solid whose surface temperature is an harmonic function of time.

We shall list the most important properties of the solution of (11) obtained:

1) The amplitude of the temperature oscillations decreases with increasing depth x according to the law

$$\exp\left[-\frac{x}{\sqrt{a}}\sqrt[4]{(\beta-k)^2+\omega^2}\cdot\cos\left(\frac{1}{2}\operatorname{arctg}\frac{\omega}{\beta-k}\right)\right],$$

i.e., the timewise damping of the temperature affects its damping with depth. From this it may also be seen that the higher the frequency ω, the faster the amplitude diminishes.



Dependence of dimensionless temperature ϑ/ϑ_0 on dimensionless time for steady temperature oscillations ($k = 0$) —dashed curves, and in the presence of damping ($k = 0.5\ \text{sec}^{-1}$)—solid curves, on the dimensionless distance from the ends of the rod $\xi = x\ \omega a =$ 0 (1), $\pi/2$ (2) and π (3).

2) The phase of the temperature wave lags according to the law

$$\frac{x}{\sqrt{a}}\sqrt[4]{(\beta-k)^2+\omega^2}\cdot\sin\left(\frac{1}{2}\operatorname{arctg}\frac{\omega}{\beta-k}\right).$$

This lag increase with increasing ω.

As an illustration we have constructed a graph of the dependence of ϑ/ϑ_0 on $\tau = \omega t$ (see figure). The thermophysical constants of the metal rods were chosen according to [4], while the temperature oscillation frequency ω was 1 sec^{-1}. Estimating the cross-sectional area of the metal rod at $10^{-3}\ \text{m}^2$, we obtain for the coefficient β a value of the order $10^{-3}\ \text{sec}^{-1}$, which may be neglected in comparison with the damping coefficient k. We note that, with the above choice of thermophysical constants, only for rods with a cross-sectional area of $S \sim 10^{-7}\ \text{m}^2$ and less, does the effect of heat transfer with the surrounding medium have an influence on the temperature distribution. Thus, in a rather wide interval of the thermophysical and dynamic parameters appearing in the solution (11) obtained, the heat-transfer coefficient β may be neglected, which simplifies the calculations considerably.

Notation

ϑ (x, t)—temperature at points at a distance x from the end surface of the rod at time t; k—temperature wave damping coefficient; ω—oscillation frequency; S—cross-sectional area of rod; P—perimeter of section; ρ—density; c—specific heat; H—heat transfer coefficient; a—thermal diffusivity.

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